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ON CONDITIONS ON THE CONTACT SURFACE OF AN ELASTIC SHELL AND AN IDEAL FLUID IN THE LAGRANGEAN REPRESENTATION

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M. A. IL'GAMOV
(Kazan')
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Two variants of the kinematic and dynamic conditions on the contact surface between an elastic shell and an ideal fluid are presented in Lagrange coordinates for separation-free and impermeable motion. The discussion of the singularities of these versions is performed for the plane problem, where the mutual slip along the contact surface is successfully taken completely into account. Furthermore, an extension of one of the versions for writing the conditions is given for the three-dimensional problem in which the mutual slip is small. The shell thickness is considered small compared to its characteristic dimensions, in which connection the contact surface is identified with the middle surface of the wall.
A formulation of the problem for both media in Lagrange variables may turn out to be most convenient for a study of the interaction between an elastic body and a fluid in certain cases. In particular, among these is the case in which the fluid and shell motion occurs primarily along the normal to the undeformed surface, and therefore, the mutual slip is relatively slight. An important example is the problem describing the process of hydrodynamic stamping of thin-walled pieces from a sheet billet.

1. Specifics of a single Lagrange representation. To illustrate the methods of writing the juncture conditions, strong bending of a beam by a force p constant in time and with a fixed line of action in space (Fig. 1) is considered. For instance, let this be the weight of a rod moving freely in rigid guides. There is no friction between the rod and the beam. During flexure, the point m of the beam with Lagrange coordinate α turns out to be under a force p, at some point in space, which had been at a point m' with the coordinate α' prior to deformation. This latter occupies a new position in space, but, however, retains the same value of the coordinate α' numerically. The force $p(\alpha')$ will also have the Lagrange coordinate α'.

The force along the normal to the cambered axis equals $p(\alpha') / \cos\beta(\alpha)$, where β is the angle between the unit vectors along the normal at the point $m(\alpha)$ before and after deformation. It is known [1] that $\cos\beta(\alpha) = 1 + \partial v(\alpha) / \partial\alpha$. Here and henceforth, $v(\alpha)$, $w(\alpha)$ and $v(\alpha')$, $w(\alpha')$ denote projections of displacements of the points $m(\alpha)$ and $m'(\alpha')$ along the axis prior to deformation. The bending equations for a distributed system of forces $p(\alpha')$ will be

$$L_i^*[v(\alpha), w(\alpha)] = \delta_{3i}\left[1 + \frac{\partial v(\alpha)}{\partial\alpha}\right]^{-1} p(\alpha') \quad (i = 2, 3) \qquad (1.1)$$

where L_2^*, L_3^* are nonlinear differential operators of the large bending equations obtained by projecting all the forces on the axis after deformation, and δ_{3i} is the Kronecker symbol.

The behavior of the system under consideration possesses the fundamental properties of interaction between the elastic body and the fluid: the magnitude of the pressure and the slip between the media depend on the bending of the beam. Similar questions, particularly the procedure for calculating the work performed by the forces "wandering" along the beam, have been discussed in [2].

The $p(\alpha')$ in the right side of (1.1) is expressed in terms of the pressure at the point $m(\alpha)$

$$p(\alpha') \approx p(\alpha) + (\alpha' - \alpha)\, \partial p(\alpha) / \partial \alpha \qquad (1.2)$$

From Fig. 1 it follows that $\alpha - \alpha' = -v(\alpha)$, hence (1.1) reduces to

$$L_i^* [v(\alpha),\ w(\alpha)] = \delta_{3i}\left[1 + \frac{\partial v(\alpha)}{\partial \alpha}\right]^{-1}\left[p(\alpha) + v(\alpha)\frac{\partial p(\alpha)}{\partial \alpha}\right] \quad (i = 2, 3) \qquad (1.3)$$

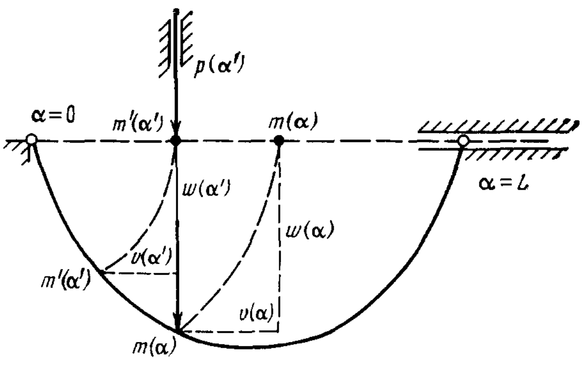

Fig. 1

The expression (1.3) remains valid until the greatest value of α satisfies the condition $\alpha < L + v(L)$, where L is the beam length. For $\alpha \geqslant L + v(L)$ the quantity in the last bracket on the right side of (1.3) should be set equal to zero (in the example under consideration $v(L) < 0$), which corresponds to the removal of part of the forces from the beam because of the displacement of its endpoint $\alpha = L$.

The boundary conditions are

$$v(\alpha) = w(\alpha) = M(\alpha) = 0 \quad (\alpha = 0) \qquad (1.4)$$

$$w(\alpha) = M(\alpha) = 0,\ N(\alpha)\cos\beta(\alpha) - Q(\alpha)\sin\beta(\alpha) = 0 \quad (\alpha = L)$$

where N, M, Q are the axial force, bending moment, and transverse force, respectively.

When the load $p(\alpha)$ is rapidly varying, and the beam displacement varies smoothly (for example, for a concentrated force), an equation in another form must be used in place of (1.3). The relation between the displacement components of the points $m(\alpha)$ and $m'(\alpha')$ can be represented in the following form to second order accuracy:

$$v(\alpha) = v(\alpha') + (\alpha - \alpha')\frac{\partial v(\alpha')}{\partial \alpha'}, \quad w(\alpha) = w(\alpha') + (\alpha - \alpha')\frac{\partial w(\alpha')}{\partial \alpha'} \tag{1.5}$$

From Fig. 1 there follows that $v(\alpha') = mm' \cos\beta(\alpha')$. Since $mm' = \alpha - \alpha'$ and $\cos\beta(\alpha') = 1 + \partial v(\alpha') / \partial \alpha'$, then

$$\alpha - \alpha' = -v(\alpha')\left[1 + \frac{\partial v(\alpha')}{\partial \alpha'}\right]^{-1} \tag{1.6}$$

Taking account of (1.6), the relationship (1.5) and any function $N(\alpha)$ can be written to the accuracy taken as:

$$v(\alpha) = v(\alpha') - v(\alpha')\frac{\partial v(\alpha')}{\partial \alpha'}, \ldots, N(\alpha) = N(\alpha') - v(\alpha')\frac{\partial N(\alpha')}{\partial \alpha'} \tag{1.7}$$

The derivation will equal

$$\frac{\partial w(\alpha)}{\partial \alpha} = \frac{\partial w(\alpha')}{\partial \alpha'} - v(\alpha')\frac{\partial^2 w(\alpha')}{\partial \alpha'^2}, \quad \frac{\partial^4 w(\alpha)}{\partial \alpha^4} = \frac{\partial^4 w(\alpha')}{\partial \alpha'^4} - v(\alpha')\frac{\partial^5 w(\alpha')}{\partial \alpha'^5}, \ldots$$

hence, the differentiation operation is performed with respect α' in the operators L_i^* of (1.1) reduced to the form

$$L_i^*\left[v(\alpha') - v(\alpha')\frac{\partial v(\alpha')}{\partial \alpha'}, \quad w(\alpha') - v(\alpha')\frac{\partial w(\alpha')}{\partial \alpha'}\right] = \delta_{3i}\left[1 + \frac{\partial v(\alpha')}{\partial \alpha'}\right]^{-1} p(\alpha') \tag{1.8}$$

Second order quantities are discarded in the brackets in the right side of (1.8). The load $p(\alpha')$ should be equated to zero for values of $\alpha' > L + v(L)$

There is no mutual slip at the fastened endpoint of the beam $(\alpha = 0)$ hence conditions (1.4) retain their previous form

$$v(\alpha') = w(\alpha') = M(\alpha') = 0 \quad (\alpha' = 0) \tag{1.9}$$

while the conditions on the sliding support $(\alpha = L)$ are written thus:

$$w(\alpha') - v(\alpha')\frac{\partial w(\alpha')}{\partial \alpha'} = 0 \tag{1.10}$$

$$M(\alpha') - v(\alpha')\frac{\partial M(\alpha')}{\partial \alpha'} = 0, \quad N(\alpha') - v(\alpha')\frac{\partial N(\alpha')}{\partial \alpha'} = 0$$

This last expression is presented for the case when the term $Q \sin \beta$ in (1.4) is small compared to the first. It is also easily written in complete form. It is important to note that conditions (1.10) should be satisfied for a variable value of $\alpha' = L + v(L)$. The discussion presented will be used later when writing the general contact conditions.

2. Kinematic and dynamic conditions on the surface. It is assumed that the geometric parameters of the shell vary smoothly, the fluid particles do not separate from its surface, and do not leave its surface because of the relative motion. Slipping between the media can be due to shell deformations, as in the example considered above, and the independent motion of the fluid. The total value of the slip should be small compared with the length on which a significant change in the deformation and pressure fields occurs in the media (for example, with the wavelength).

At time $t = 0$ the orthogonal curvilinear system of Lagrange coordinates α_1, α_2, α_3 is assumed common for the elastic body and the domain occupied by the fluid. Contact between them is realized along the surface $\alpha_3 = \alpha_3^\circ$ (Fig. 2). The coordinates α_1, α_2 are directed along the lines of principal curvature of the surface. In particular, the nondeformed state of the elastic body (as in the previous example) can be taken as the time $t = 0$. Therefore, the same Lagrange coordinates are ascribed to adjacent points on both sides of the interfacial surface α_3° at the initial moment. Although these material points later diverge while remaining on the same surface α_3° and move along different trajectories, the value of their coordinates will be numerically identical. Hence, the contact conditions in the presence of slip will be compiled for points having different values of the Lagrange coordinates on the contact surface, for instance, $(\alpha_1, \alpha_2, \alpha_3^\circ)$ and $(\alpha_1', \alpha_2', \alpha_3^\circ)$.

Thus the point $m\ (\alpha_1, \alpha_2, \alpha_3^\circ)$ of the elastic body coincides at the time t with the fluid particle M', on the same surface α_3°, but having the coordinates α_1', α_2' for $t = 0$ (Fig. 2). The new positions of the points $m'\ (\alpha_1', \alpha_2', \alpha_3^\circ)$ and $M\ (\alpha_1, \alpha_2, \alpha_3^\circ)$ are also shown in the Figure. The kinematic condition will be [3]

$$\boldsymbol{\rho}(\alpha_1, \alpha_2, \alpha_3^\circ) + \mathbf{u}(\alpha_1, \alpha_2, \alpha_3^\circ, t) = \boldsymbol{\rho}(\alpha'_1, \alpha_2', \alpha_3^\circ) + \mathbf{U}(\alpha_1', \alpha_2', \alpha_3^\circ, t) \tag{2.1}$$

Here $\boldsymbol{\rho}$ is a radius-vector, and $\mathbf{u}$, $\mathbf{U}$ are displacement vectors of particles of the elastic body and the fluid.

In the case of an ideal fluid the dynamic conditions have the form

$$\sigma_{k_3}(\alpha_1, \alpha_2, \alpha_3^\circ, t)\cdot \mathbf{k}_i^*(\alpha_1, \alpha_2, t) = 0 \quad (i = 1, 2) \tag{2.2}$$

$$\sigma_{k_3}(\alpha_1, \alpha_2, \alpha_3^\circ, t)\cdot \mathbf{n}_3^*(\alpha_1, \alpha_2, t) = -p(\alpha_1', \alpha_2', \alpha_3^\circ, t)$$

where σ_{k_3} is the stress vector on an area which had the normal $\mathbf{k}_3$ prior to deformation; $\mathbf{k}_i{}^*$ are unit vectors along the tangents to the coordinate lines α_i at the point m after deformation, and $\mathbf{n}_3{}^*$ is the unit vector along the normal. The pressure in the fluid p is related to the density and the displacement vector components U_1, U_2, U_3 by means of the motion and continuity equations. The relationships between the unit directions $\mathbf{k}_i$, $\mathbf{k}_j{}^*$, $\mathbf{n}_3{}^*$ contain the displacement components of the elastic body u_1, u_2, u_3 and their derivatives [4].

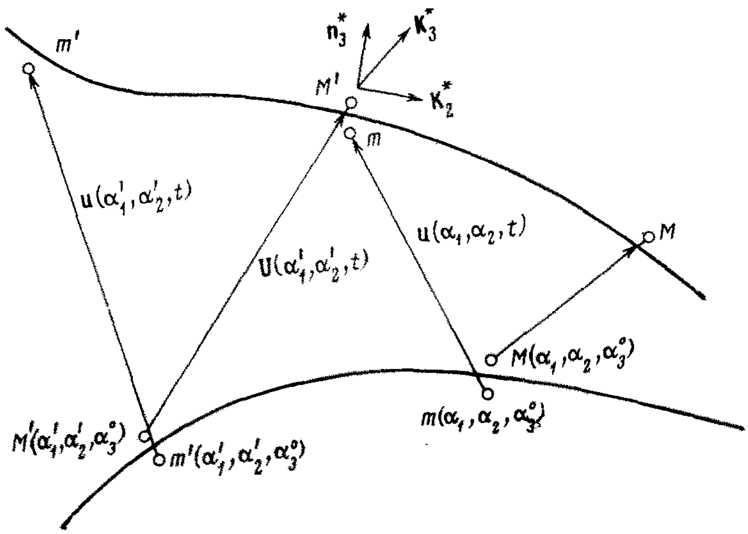

Fig. 2

If the elastic body is a thin-walled shell, its middle surface can be taken as the contact surface $\alpha_3 = \alpha_3{}^\circ$. Hence, (2.1) is not changed, but the equations

$$L_i{}^* \left[u_1(\alpha_1, \alpha_2, t),\ u_2, u_3\right] = \delta_{3i} p\,(\alpha_1', \alpha_2', \alpha_3{}^\circ, t) \quad (i = 1, 2, 3) \tag{2.3}$$

will replace (2.2) where $L_1{}^*$, $L_2{}^*$, $L_3{}^*$ are nonlinear differential operators of the theory of shells which correspond to forming the equations of motion by summing the projections of all the forces on the coordinate axes after deformation [1]. The displacements can be on the order of the characteristic dimension of the shell, and the deformations are assumed small compared with one. According to the Kirchhoff-Love hypothesis $\mathbf{k}_3{}^* = \mathbf{n}_3{}^*$.

The support conditions

$$l_\nu \left[u_1(\alpha_1{}^\circ, \alpha_2, t),\ u_2, u_3\right] = 0 \quad (\nu = 1, 2, 3, 4) \tag{2.4}$$

are given the boundary sections of the shell.

The right sides in the conditions (2.1)-(2.3) can be expressed in terms of functions defined in the neighborhood of the point $m\ (\alpha_1,\ \alpha_2,\ \alpha_3^\circ)$ with the local trihedron $\mathbf{k}_i\ (\alpha_1,\ \alpha_2)$ prior to deformation. The components $u_i,\ U_i$ are hence introduced in the form

$$\mathbf{u} = \mathbf{k}_i\,(\alpha_1,\ \alpha_2)u_i\,(\alpha_1,\ \alpha_2,\ \alpha_3^\circ,\ t),\quad \mathbf{U} = \mathbf{k}_i(\alpha_1,\ \alpha_2)U_i\,(\alpha_1,\ \alpha_2,\ \alpha_3^\circ,\ t) \tag{2.5}$$

Another version of the relationships on the contact surface is obtained if the left sides of (2.1)-(2.4) are expressed in terms of functions given at the point $m'\ (\alpha_1',\ \alpha_2',\ \alpha_3^\circ)$. In this case

$$\mathbf{u} = \mathbf{k}_i\ (\alpha_1',\ \alpha_2')u_i\,(\alpha_1',\ \alpha_2',\ \alpha_3^\circ,\ t),\quad \mathbf{U} = \mathbf{k}_i\ (\alpha_1',\ \alpha_2')U_i\ (\alpha_1',\ \alpha_2',\ \alpha_3^\circ,\ t) \tag{2.6}$$

i.e., $u_i,\ U_i$ will be displacement projections of the same particle of the elastic body and the fluid as above in the directions of the unit vectors $\mathbf{k}_i\ (\alpha_1',\ \alpha_2')$ at the point $m'\ (\alpha_1',\ \alpha_2',\ \alpha_3^\circ)$.

The features of these two versions for writing the conditions are discussed later for the problem in the $\alpha_2\ \alpha_3$ plane, after which a generalization of the necessary relationships for the three-dimensional problem is given.

3. First version of the conditions. The functions ρ and $\mathbf{U}$ are assumed analytic in α_2, whereupon the right side in (2.1) can be expanded in a Taylor series in the neighborhood of the point $m\ (\alpha_2,\ \alpha_3^\circ)$. If three terms are retained in this expansion, and the Fresnet formula is used, then it follows from (2.1) that

$$\mathbf{U}\,(\alpha_2,\ t) - \mathbf{u}\,(\alpha_2,\ t) + (\alpha_2' - \alpha_2)\left(h_2\mathbf{k}_2 + \frac{\partial \mathbf{U}}{\partial \alpha_2}\right) + \frac{1}{2}(\alpha_2' - \alpha_2)^2\left(\frac{\partial h_2}{\partial \alpha_2}\,\mathbf{k}_2 - \frac{h_2\partial h_2}{h_3\partial \alpha_3}\,\mathbf{k}_3 + \frac{\partial^2 \mathbf{U}}{\partial \alpha_2^2}\right) = 0 \tag{3.1}$$

Here $h_2,\ h_3$ are Lamé coefficients. The two scalar equations obtained from (3.1) and (2.5) by subtracting one from the other and by adding can be reduced to linear and quadratic equations in $(\alpha_2' - \alpha_2)$. The first yields

$$\alpha_2' - \alpha_2 = \Delta\ (\alpha_2,\ t),\quad \Theta_{23} = {}^1/_2 E_{23} + \Omega_1 \tag{3.2}$$

$$\Delta\,(\alpha_2,\ t) = \frac{1}{h_2}\ \frac{(u_2 - U_2)F + (u_3 - U_3)}{(1 + E_{22})F + \Theta_{23}}$$

$$F = \left[\frac{h_2\partial h_2}{h_3\partial \alpha_3}\,(1 + E_{22}) - \frac{\partial}{\partial \alpha_2}(h_2\Theta_{23})\right]\left[\frac{\partial}{\partial \alpha_2}\,(h_2 + h_2 E_{22}) + \frac{h_2\partial h_2}{h_3\partial \alpha_3}\Theta_{23}\right]^{-1};$$

The parameters $E_{22},\ E_{23},\ \Omega_1,$ characterizing the deformation and mean rotation of a volume element of the fluid, are analogous to the corresponding parameters in the theory of elasticity [4]

$$E_{22} = \frac{\partial U_2}{h_2\partial \alpha_2} + \frac{\partial h_2}{h_2 h_3 \partial \alpha_3}\,U_3,\quad E_{23} = \frac{h_3\partial}{h_2\partial \alpha_2}\left(\frac{U_3}{h_3}\right) + \frac{h_2\partial}{h_3\partial \alpha_3}\left(\frac{U_2}{h_2}\right) \tag{3.3}$$

$$2\Omega_1 = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial\alpha_2}(h_3U_3) - \frac{\partial}{\partial\alpha_3}(h_2U_2)\right]$$

After elimination of $(\alpha_2' - \alpha_2)$ from the quadratic equation, the kinematic condition on the contact surface between an elastic body and an ideal fluid takes the form

$$[(u_3 - U_3)(1 + E_{22}) - (u_2 - U_2)\Theta_{23}][(1 + E_{22})F + \Theta_{23}] + \frac{1}{2h_2^2}[u_3 - U_3 + (u_2 - U_2)F]^2\left[\frac{\partial}{\partial\alpha_2}(h_2 + h_2E_{22}) + \frac{h_2\partial h_2}{h_3\partial\alpha_3}\Theta_{23}\right] = 0 \quad (3.4)$$

The right sides in (2.2) and (2.3) are also expanded in a Taylor series in the neighborhood of the point $m(\alpha_2, \alpha_3^\circ)$, and the difference $(\alpha_2' - \alpha_2)$ is replaced by the relationship (3.2). In the case of a thin-walled shell the dynamic conditions will therefore have the form

$$L_i^*[u_2(\alpha_2, t), u_3] = \delta_{3i}\left[p(\alpha_2, \alpha_3^\circ, t) + \Delta(\alpha_2, t)\frac{\partial p}{\partial\alpha_2} + \frac{\Delta^2}{2}\frac{\partial^2 p}{\partial\alpha_2^2}\right] \quad (\alpha_3 = \alpha_3^\circ) \quad (3.5)$$

If there is no mutual slip between the media in contact $(u_2 = U_2)$, then according to (3.4) and (3.5)

$$u_3 = U_3, \quad L_i^*[u_2(\alpha_2, t), u_3] = \delta_{3i}p(\alpha_2, \alpha_3^\circ, t) \quad (3.6.)$$

Therefore, for arbitrary but mutually equal shell and fluid displacements along the tangential coordinate, the kinematic condition in linear independently of the magnitude of the normal displacements, in contrast to the cases of the Euler and the mixed Euler-Lagrange approaches.

The intermediate case between conditions (3.4)-(3.6) holds when the inequality

$$|u_2 - U_2| \leqslant 2h_2^2(1 + E_{22})^2\left[\frac{\partial}{\partial\alpha_2}(h_2 + h_2E_{22}) + \frac{h_2\partial h_2}{h_3\partial\alpha_3}\Theta_{23}\right]^{-1}$$

which corresponds to retaining the first two members of the expansion in (3.1), is satisfied. The kinematic and dynamic conditions hence become

$$(u_3 - U_3)\left(1 + \frac{\partial U_2}{h_2\partial\alpha_2} + \frac{\partial h_2}{h_2h_3\partial\alpha_3}U_3\right) + (u_2 - U_2)\left(\frac{\partial h_2}{h_2h_3\partial\alpha_3}U_2 - \frac{\partial U_3}{h_2\partial\alpha_2}\right) = 0 \quad (3.7)$$

$$L_i^*[u_2, u_3] = \delta_{3i}p(\alpha_2, \alpha_3^\circ, t) + \frac{\delta_{3i}\partial p}{h_2\partial\alpha_2}\left(1 + \frac{\partial U_2}{h_2\partial\alpha_2} + \frac{\partial h_2}{h_2h_3\partial\alpha_3}U_3\right)^{-1}(u_2 - U_2) \quad (3.8)$$

These last conditions are obtained in [3] for the three-dimensional problem. In a linear approximation, conditions (3.4), (3.5) as well as (3.7), (3.8) agree with the corresponding conditions written in the mixed Euler-Lagrange form. Velocity components figure in the latter in place of the displacement components. However, the linearized expressions (3.4), (3.7) and the right sides (3.5), (3.8) differ from the analogous conditions in mixed form. If φ° and φ denote the velocity potential in a fluid for an absolutely rigid boundary $\alpha_3 = \alpha_3^\circ$ $(\partial\varphi^\circ/\partial\alpha_3 = 0)$ and its perturbation because of boundary displacement by u_2, u_3, then the kinematic condition linearized with respect to $\varphi,\ u_2,\ u_3$ in mixed form is [5]

$$\frac{\partial u_n}{\partial t} = \frac{\partial \varphi}{h_3 \partial x_3} - \omega_2 \frac{\partial \varphi^\circ}{h_2 \partial x_2} + \frac{u_3}{h_3}\left(\frac{\partial^2 \varphi^\circ}{h_3 \partial x_3^2} + \frac{\partial h_3}{h_2 \partial x_2}\frac{\partial \varphi^\circ}{h_2 \partial x_2}\right) - \frac{u_2 \partial h_2}{h_2 h_3 \partial x_3}\frac{\partial \varphi^\circ}{h_2 \partial x_2} \tag{3.9}$$

where x_i are Euler coordinates which can here be identified with the Lagrange coordinates α_i and ω_n is the angle of rotation of a shell section along the coordinate line α_2.

Representing the total displacement and pressure in the form

$$U_i^* = U_i^\circ + U_i, \quad u_i^* = u_i, \quad p^* = p^\circ + p \quad (i = 2, 3)$$

we find the linearized condition $(U_3^\circ = \partial U_3^\circ / \partial \alpha_2 = 0$ на $\alpha_3 = \alpha_3^\circ)$ from (3.7):

$$\left(1 + \frac{\partial U_2^\circ}{h_2 \partial \alpha_2}\right)(u_3 - U_3) - \frac{\partial h_2}{h_2 h_3 \partial \alpha_3} U_2^\circ (u_2 - U_2) - U_2^\circ\left(\frac{\partial h_2}{h_2 h_3 \partial \alpha_3} U_2 - \frac{\partial U_3}{h_2 \partial \alpha_2}\right) = 0 \tag{3.10}$$

Condition (3.10) and the linearized right side of (3.8) differ from (3.9) and the linearized right side of the dynamic conditions of mixed type in structure. Moreover derivatives of the displacement and pressure components in the fluid with respect to the coordinate figure in (3.7), (3.8), (3.10) and components of the shell displacement are in the form of the functions themselves. In certain problems this circumstance can be considered a disadvantage of this variant in writing and the conditions. For example, if the field in the fluid is rapidly varying (high-frequency vibrations, shocks, etc.) and it is determined approximately, then a loss of accuracy occurs because of differentiation. The second variant in the writing of the conditions. For example, if the field in the fluid is rapidly varying (high-frequency vibrations, shocks, etc.) and it is determined approximately, then a loss of accuracy occurs because of differentiation. The second variant in the writing of the conditions is free of this disadvantage.

4. Second variant of the conditions. Expansion of the left side of the vector equation (2.1) in a Taylor series in the neighborhood of the point m' with coordinates $\alpha_2',\ \alpha_3''$ (Fig. 2) yields

$$\alpha_2 - \alpha_2' = \Delta(\alpha_2', t), \quad \theta_{23} = {}^1/_2\, e_{23} + \omega_1 \tag{4.1}$$

$$\Delta(\alpha_2', t) = \frac{1}{h_2}\frac{(U_2 - u_2) f + (U_3 - u_3)}{(1 + e_{22}) f + \theta_{23}}$$

$$f = \left[\frac{h_2 \partial h_2}{h_3 \partial \alpha_3}(1 + e_{22}) - \frac{\partial}{\partial \alpha_2'}(h_2 \theta_{23})\right]\left[\frac{\partial}{\partial \alpha_2'}(h_2 + h_2 e_{22}) + \frac{h_2 \partial h_2}{h_3 \partial \alpha_3}\theta_{23}\right]^{-1}$$

The representation (2.6) has been used here. The quantities e_{22}, e_{23} are obtained from (3.3) by replacing U_i and α_2 by u_i and α_2'. In the case of a thin-walled shell $h_3 = 1$. $\partial h_2 / \partial \alpha_3 \approx h_2 k_{22}$ where k_{22} is the curvature of the middle surface and the quantity $h_2(\alpha_3)$ can be taken as the Lamé coefficient for the middle surface to the accuracy of hk_{22} (h is the shell thickness) as compared to one. The quantity θ_{23} hence agrees with ω_2 in (3.9).

In the second variant, the displacement and pressure components are functions of the argument α_2'; derivatives are also taken with respect to α_2'. The quantities $\Delta(\alpha_2, t)$, and $\Delta(\alpha_2', t)$ are different functions. For instance, the relationships

$\alpha_2 - \alpha_2' = -u_2(\alpha_2)$ and (1.6), which were used in Sect. 1, follow from (3.2) and (4.1) in a Cartesian system and for $U_2 \equiv 0$. The denominator and numerator in (3.2) and (4.1) must hence be divided by F and f.
The kinematic condition of the second variant has the form

$$[(U_3 - u_3)(1 + e_{22}) - (U_2 - u_2)\theta_{23}][(1 + e_{22})f + \theta_{23}] + \frac{1}{2h_2^2} \times [U_3 - u_3 + (U_2 - u_2)f]^2 \left[\frac{\partial}{\partial\alpha_2'}(h_2 + h_2 e_{22}) + \frac{h_2 \partial h_2}{h_3 \partial\alpha_3}\theta_{23}\right] = 0 \quad (\alpha_3 = \alpha_3^\circ) \tag{4.2}$$

Representing (4.1) as $\alpha_2 = \alpha_2' + \Delta(\alpha_2', t)$, we will have the following dynamical conditions instead of (2.3) and (2.4):

$$L_i^*\left[u_2(\alpha_2', t) + \Delta(\alpha_2', t)\frac{\partial u_2}{\partial\alpha_2'} + \frac{\Delta^2}{2}\frac{\partial^2 u_2}{\partial\alpha_2'^2},\quad u_3(\alpha_2', t) + \ldots\right] = \delta_{3i}p(\alpha_2', \alpha_3^\circ, t) \tag{4.3}$$

$$l_\nu\left[u_2(\alpha_2', t) + \Delta\frac{\partial u_2}{\partial\alpha_2'} + \frac{\Delta^2}{2}\frac{\partial^2 u_2}{\partial\alpha_2'^2},\ u_3(\alpha_2', t) + \ldots\right] = 0 \ (\nu = 2, 3, 4) \tag{4.4}$$

Condition (4.4) should be satisfied for a variable value of $\alpha_2' = \alpha_2^\circ + \Delta(\alpha_2^\circ, t)$, where α_2° is the coordinate of the boundary section of the shell. If the support conditions $u_2(\alpha_2, t) = u_3(\alpha_2, t) = 0, \ldots (\alpha_2 = \alpha_2^\circ)$, have been posed on the shell edges for the equation of motion (3.5) in the first variant, for example, then the now become

$$u_\nu(\alpha_2, t) = u_\nu(\alpha_2', t) + \Delta(\alpha_2', t)\frac{\partial u_\nu}{\partial\alpha_2'} + \frac{\Delta^2}{2}\frac{\partial^2 u_\nu}{\partial\alpha_2'^2} = 0 \quad (\alpha_2' = \alpha_2^\circ + \Delta(\alpha_2^\circ, t))$$

where $\Delta(\alpha_2^\circ, t)$ will depend only on the fluid parameters because $u_2(\alpha_2^\circ, t) = u_3(\alpha_2^\circ, t) = 0$ since

$$\Delta(\alpha_2^\circ, t) = -\frac{1}{h_2}\frac{U_2 F + U_3}{(1 + E_{22})F + \theta_{23}}$$

Therefore, in contrast to the first variant of writing the conditions, as is seen from (4.1)-(4.4) there are no derivatives of the fluid parameters U_i, p, in the second variant, which may be considered its advantage. However, the equations of shell motion and the boundary conditions on its section (4.3) and (4.4) turn out to be more complex than in the first variant. Hence, the question of which is more preferable should be resolved in an examination of specific problems. Conditions of the first variant admit of simplification in the consideration of particular kinds of fluid motion and its physical properties, while conditions of the second variant simplify in the examination of different kinds of shell deformation.
According to (4.1), (4.2) we have $U_3 = u_3$, $\Delta = 0$, for $U_2 = u_2$ and both variants of writing the conditions are identical in form.
If we limit ourselves to two terms in the Taylor series, the kinematic condition (4.2) is

written as follows:

$$(U_3 - u_3)\left(1 + \frac{\partial u_2}{h_2\partial\alpha_2'} + \frac{\partial h_2}{h_2h_3\partial\alpha_3}u_3\right) + (U_2 - u_2)\left(\frac{\partial h_2}{h_2h_3\partial\alpha_3}u_2 - \frac{\partial u_3}{h_2\partial\alpha_2'}\right) = 0 \tag{4.5}$$

Only terms containing $\Delta(\alpha_2', t)$, $\Delta(\alpha_2^\circ, t)$ to the first degree should hence be retained in (4.3), (4.4). The latter are:

$$\Delta(\alpha_2', t) = \frac{U_2 - u_2}{h_2}\left(1 + \frac{\partial u_2}{h_2\partial\alpha_2'} + \frac{\partial h_2}{h_2h_3\partial\alpha_3}u_3\right)^{-1}, \quad \Delta(\alpha_2^\circ, t) = -\frac{U_2}{h_2}\left(1 + \frac{\partial U_2}{h_2\partial\alpha_2} + \frac{\partial h_2}{h_2h_3\partial\alpha_3}U_3\right)^{-1} \tag{4.6}$$

In contrast to (3.10), the linearized condition (4.5)

$$U_3 = u_3 + U_2^\circ\left(\frac{\partial u_3}{h_2\partial\alpha_2'} - \frac{\partial h_2}{h_2h_3\partial\alpha_3}u_2\right) \quad (\alpha_3 = \alpha_3^\circ) \tag{4.7}$$

is similar in structure to (3.9). In the case of a flat plate and a homogeneous initial field in the fluid $(U_2^\circ = U_2^\circ(t),\ U_3 = 0)$ they agree in form if the displacement components in (4.7) are replaced by the corresponding velocity components. It is evidently impossible to identify α_2' and $x_2 \approx \alpha_2$ in this case.

The displacement U_2^0 along α_2' will be constant, say, for reciprocating motion of an ideal incompressible fluid along the boundary of a half-space. The maximum fluid displacement should hence satisfy the constraint mentioned in Sect. 2 for the presented equations to be applicable. If there is flow with constant velocity, then it is necessary to limit the time during which the process is considered. In this latter case the problem should be formulated in mixed form for an arbitrary time interval [5].

According to (4.6), the functions $\Delta(\alpha_2', t)$ and $\Delta(\alpha_2^\circ, t)$ in the dynamic conditions (4.3), (4.4), will have the form:

$$\Delta(\alpha_2', t) = \frac{U_2^\circ}{h_2}, \quad \Delta(\alpha_2^\circ, t) = -\frac{U_2^\circ}{h_2}\left(1 + \frac{\partial U_2^\circ}{h_2\partial\alpha_2}\right)^{-1} \tag{4.8}$$

In the case of a homogeneous field in the fluid $\Delta(\alpha_2', t) = -\Delta(\alpha_2^\circ, t) = U_2^\circ/h_2$.

As an illustration, let us reduce the equation

$$D\frac{\partial^4 w}{\partial x^4} - N\frac{\partial^2 w}{\partial x^2} + \gamma h\frac{\partial^2 w}{\partial t^2} = p^*(x', t), \quad N = \frac{B}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx$$

to the form it should have in the second variant. Here $x = h_2\alpha_2$, $x' = h_2\alpha_2'$. If we limit ourselves to the case (4.7), (4.8) and take into account that N is independent of the longitudinal coordinate, then

$$D\left(\frac{\partial^4 w}{\partial x'^4} + U_2^0 \frac{\partial^5 w}{\partial x'^5}\right) - N\left(\frac{\partial^2 w}{\partial x'^2} + U_2^0 \frac{\partial^3 w}{\partial x'^3}\right) + \gamma h \frac{\partial^2}{\partial t^2}\left(w + U_2^0 \frac{\partial w}{\partial x'}\right) = p^*(x', t)$$

The first equation is valid in the interval $0 < x < L$ and the second for $U_2^0 < x' < L - U_2^0$.

5. Second variant of the conditions in the three-dimensional case. If only two terms are taken into account in the Taylor series, the kinematic condition is

$$(u_1 - U_1)[(^1/_2 e_{12} + \omega_3)(^1/_2 e_{23} + \omega_1) - (1 + e_{22})(^1/_2 e_{13} - \omega_2)] + (u_2 - U_2)[(^1/_2 e_{12} - \omega_3)(^1/_2 e_{13} - \omega_2) - (1 + e_{11})(^1/_2 e_{23} + \omega_1)] + (u_3 - U_3)[(1 + e_{11})(1 + e_{22}) - (^1/_4 e_{12}^2 - \omega_3^2)] = 0 \quad (\alpha_3 = \alpha_3^\circ) \tag{5.1}$$

The expressions for $e_{ij}\ (i \neq j),\ \omega_k$ are obtained from (3.3) by replacing U_i, α_1, α_2 by $u_i, \alpha_1', \alpha_2'$, and the cyclic permutation for e_{ii} from

$$e_{11} = \frac{\partial u_1}{h_1 \partial \alpha_1'} + \frac{\partial h_1}{h_1 h_2 \partial \alpha_2'} u_2 + \frac{\partial h_1}{h_1 h_2 \partial \alpha_3} u_3$$

The relation between α_1, α_2 and α_1', α_2' is the following

$$\alpha_1 = \alpha_1' + \Delta_1(\rho'), \quad \Delta_2 = \alpha_2' + \Delta_2(\rho') \tag{5.2}$$

$$\Delta_1(\rho') = \frac{1}{h_1} \frac{(U_1 - u_1)(1 + e_{22}) - (U_2 - u_2)(^1/_2 e_{12} - \omega_3)}{(1 + e_{11})(1 + e_{22}) - (^1/_4 e_{12}^2 - \omega_3^2)}$$

Here and henceforth, (ρ') and (ρ) denote the arguments $(\alpha_1', \alpha_2', t)$ and (α_1, α_2, t). The expression for $\Delta_2(\rho')$ is obtained from $\Delta_1(\rho')$ by mutual replacement of the subscripts 1 and 2 and also by replacing ω_3 by $-\omega_3$. We have in the relationships $\alpha_i' = \alpha_i + \Delta_i(\rho)$

$$\Delta_1(\rho) = \frac{1}{h_1} \frac{(u_1 - U_1)(1 + E_{22}) - (u_2 - U_2)(^1/_2 E_{12} - \Omega_3)}{(1 + E_{11})(1 + E_{22}) - (^1/_4 E_{12}^2 - \Omega_3^2)} \tag{5.3}$$

Hence $\Delta_2(\rho)$ is obtained by using the same replacement as for $\Delta_2(\rho')$.

The dynamical conditions (2.2) are in vector form

$$\{\sigma_{k_3}(\rho') + [\boldsymbol{\Delta}(\rho') \cdot \nabla]\sigma_{k_3}\}\mathbf{k}_i^*(\rho') + \sigma_{k_3}[\boldsymbol{\Delta}(\rho') \cdot \nabla]\mathbf{k}_i^*(\rho') = 0 \quad (i = 1, 2) \tag{5.4}$$

$$\{\sigma_{k_3}(\rho') + [\boldsymbol{\Delta}(\rho') \cdot \nabla]\sigma_{k_3}\}\mathbf{n}_3^*(\rho') + \sigma_{k_3}[\boldsymbol{\Delta}(\rho') \cdot \nabla]\mathbf{n}_3^*(\rho') = -p(\rho')$$

Here ∇ is the Hamilton operator in the coordinates α_1', α_2'. The expressions (5.4) can be written in the unit vectors $\mathbf{k}_i$ prior to deformation by means of known formulas relating $\mathbf{k}_i^*$, $\mathbf{n}_3^*$, $\mathbf{k}_i$.

The expressions e_{ij}, ω_k are simplified correspondingly in the consideration of a thin-walled shell, therefore (5.1), (5.2) are also simplified somewhat. The dynamical conditions (2.3), (2.4) are written in the form

$$L_i^* \{u_1(\rho') + [\Delta(\rho')\cdot\nabla] u_1(\rho'), u_2, u_3\} = \delta_{3i} p(\rho')$$
$$l_\nu \{u_1(\rho') + [\Delta(\rho')\cdot\nabla] u_1(\rho'), u_2, u_3\} = 0 \quad [\alpha_1' = \alpha_1^\circ + \Delta_1(\alpha_1^\circ, \alpha_2, t)] \tag{5.5}$$

Terms containing $\Delta_1(\rho')$ and $\Delta_2(\rho')$ in just the first degree should be taken into account in addition to the fundamental terms in (5.5).

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